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Pancharatnam phase for displaced number states

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Abstract. The Pancharatnam phase for displaced number states of the harmonic oscillator is discussed. In particular, it is examined how a single quantum oscillator, driven by a suitable transient external force, evolves from the initial eigenstate $u_m(x)$ to the final, displaced number state *modified with a suitable phase factor*. The significance of this, usually neglected, phase factor for the solution of the relevant time-dependent Schrödinger equation and for the geometric phase accumulated in the wavefunction during the time evolution of the system is examined. The general expression for the geometric phase for a *non-cyclic* evolution from an initial displaced number state at time t_1 to the final state at time t_2 is derived and two applications, that of δ and harmonic forcing, are worked out. The special case of cyclic evolution is subsequently discussed and, in particular, the conditions leading to such an evolution are derived. The relationship to the classical notion of cyclic evolution is also examined in some detail and it is demonstrated that the general expression for the Pancharatnam phase reduces to the corresponding Berry phase. It is found that, in the case of cyclic evolution, the geometric phase for displaced number states becomes independent of the quantum number m of the initial eigenstsate, and becomes equal to the geometric phase for the coherent states. The general considerations are illustrated with the special case of the harmonic forcing function. Finally, the possibility of experimental verification, in the realm of quantum optics, is briefly considered.

1. Introduction

Since the discovery by Berry [1] of the general existence of an observable phase accumulation in the wavefunction of a quantum-mechanical system with an adiabatically changing Hamiltonian, our understanding of this phase has greatly increased. The Berry phase has attracted much theoretical interest [2–8] and it has been repeatedly corroborated by experiment [9–16]. The restriction to adiabaticity was lifted by Aharonov and Anadan [17] by removing from the wavefunction the time integral of the expectation value of the Hamiltonian as a *dynamical phase*. It was shown that once the dynamical phase is removed, the phase difference accumulated during the time evolution of the system has a purely geometric origin. A recent review of the calculation of the non-adiabatic Berry phases is given by Moore [18]. Finally, the restriction to cyclic motion and unitary evolution was removed by Samuel and Bhandari [19]. Their work was based on the earlier investigation of Pancharatnam [20] on the interference of polarized light.

The geometric (topological) phase for the case of the forced quantum oscillator with, or without, time-dependent mass and frequency has been discussed repeatedly in this journal [21–26] and elsewhere [27, 28]. Various approaches to the calculation of the geometric phase have been explored [18] and it has been demonstrated that the *coherent* states provide an especially convenient tool. In this paper, we examine the Pancharatnam phase [18, 19] for *displaced number states* of the harmonic oscillator [29–32]. The importance and unusual properties of these states have been discussed recently in some detail in [33, 34]. It is

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hoped that these states can be prepared, in the realm of quantum optics, by driving the microwave cavity field (initially prepared in a number state) of the micromaser with a classical current [33]. In section 2 we review the properties of displaced number states relevant for the present work. In section 3 we examine how a single quantum oscillator, driven by a suitable transient external force, evolves from the initial eigenstate $u_m(x)$ to the final, displaced number state *modified with a suitable phase factor*. This problem was solved some time ago and in several different ways [35–44]. We present an exact and straightforward solution of the relevant time-dependent Schrödinger equation and discuss the significance of the 'unimportant' phase factor which has been, as a rule, simply ignored. In section 4 we derive the geometric phase for generally *non-cyclic* evolution from an initial displaced number state at time t_1 to the final state at time t_2. Two simple examples, that of δ and harmonic forcing, are then presented. In section 5, the special case of cyclic evolution is discussed. We demonstrate that the general expression for the Pancharatnam phase reduces to the corresponding Berry phase and, additionally, examine in some detail the cyclic evolution for the special case of the harmonic forcing function. Finally, in section 6 we present our conclusions.

2. Displaced number states

In this section we review the properties of displaced number states relevant for the later work and fix the notation. When in the general solution of the time-dependent Schrödinger equation for a harmonic oscillator

$$\Psi(x,t)=\sum_{n=0}^{\infty}c_n\exp\left(-\frac{\mathrm{i}}{\hbar}E_nt\right)u_n(x) \tag{2.1}$$

we take [29, 31–34]

$$c_n\to c_n^{(m)}\equiv\left(\frac{m!}{n!}\right)^{1/2}\alpha^{n-m}\exp\left(-\frac{|\alpha|^2}{2}\right)L_m^{(n-m)}(|\alpha|^2) \tag{2.2}$$

with $m=0,1,2,\ldots$, we obtain a normalized coordinate-space wavefunction representing the displaced number state $\Psi_\alpha^{(m)}(x,t)$. In (2.1), $u_n(x)$ denotes the harmonic oscillator eigenfunction corresponding to the nth energy level E_n [45], while in (2.2)

$$\alpha\equiv\alpha_1+\mathrm{i}\alpha_2=|\alpha|\exp(\mathrm{i}\varphi) \tag{2.3}$$

denotes the complex displacement parameter and $L_m^{(k)}$ the associated Laguerre polynomial [46]. With the help of the generating function for the product of the associated Laguerre and Hermite polynomials it is possible to sum (2.1) and thus to obtain the time-dependent wavefunction for the displaced number state in a closed form [47]:

$$\Psi_\alpha^{(m)}(x,t)=\exp(\mathrm{i}\phi_m)u_m(x-x_c(t)). \tag{2.4}$$

Here, the phase ϕ_m is given by

$$\phi_m(x,t)=-(m+\tfrac{1}{2})\omega t+\frac{|\alpha|^2}{2}\sin 2(\omega t-\varphi)+\frac{xp_c(t)}{\hbar} \tag{2.5}$$

with

$$x_c(t) = \langle x \rangle = 2\sigma|\alpha|\cos(\omega t - \varphi) \tag{2.6}$$

$$p_c(t) = \langle p \rangle = -\frac{\hbar|\alpha|}{\sigma}\sin(\omega t - \varphi) \tag{2.7}$$

and

$$\sigma \equiv \left(\frac{\hbar}{2\mu\omega}\right)^{1/2} \tag{2.8}$$

denoting the half-width of the harmonic oscillator ground state $u_0(x)$. (In (2.8), μ represents the oscillator mass.) For $m = 0$, $\Psi_\alpha^{(m)}(x,t)$ reduces to the familiar *coherent* state wavefunction. We shall also need the corresponding momentum space wavefunction

$$\Phi_\alpha^{(m)}(p,t) = \exp(i\phi'_m)v_m(p - p_c(t)). \tag{2.9}$$

Here, $v_m(p)$ denotes the momentum-space eigenfunction corresponding to $u_m(x)$ [48]. The phase ϕ'_m is given by

$$\phi'_m(p,t) = -(m + \tfrac{1}{2})\omega t - \frac{|\alpha|^2}{2}\sin 2(\omega t - \varphi) - \frac{x_c(t)p}{\hbar}. \tag{2.10}$$

Displaced number states have a number of interesting properties. They are not mutually orthogonal:

$$\int_{-\infty}^{+\infty} \Psi_\alpha^{(m)*}(x,t)\Psi_{\alpha'}^{(m)}(x,t')\,dx = L_m^{(0)}(|\alpha e^{-i\omega t} - \alpha' e^{-i\omega t'}|^2)$$
$$\times \exp\left(-\frac{|\alpha|^2}{2} - \frac{|\alpha'|^2}{2} + \alpha^*\alpha' \exp[i\omega(t-t')] + i(m + \tfrac{1}{2})\omega(t-t')\right). \tag{2.11}$$

The proof of (2.11) is based on the following relation (α, β are complex):

$$e^{-\alpha^*\beta}\sum_{n=0}^{\infty}\left(-\frac{\alpha^*}{\beta^*}\right)^{n-m} L_m^{(n-m)}(|\alpha|^2)L_n^{(m-n)}(|\beta|^2) = L_m^{(0)}(|\alpha - \beta|^2). \tag{2.12}$$

The set $\{\Psi_\alpha^{(m)}(x,t)\}$, for any $m = \text{const}$, is (over)complete:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} d\alpha_1 \int_{-\infty}^{+\infty} d\alpha_2\, \Psi_\alpha^{(m)}(x,t)\Psi_\alpha^{(m)*}(x',t) = \delta(x - x'). \tag{2.13}$$

We also mention that

$$\int_{-\infty}^{+\infty} \Psi_\alpha^{(m)*}(x,t)\Psi_\alpha^{(m')}(x,t)\,dx = \delta_{mm'}. \tag{2.14}$$

The form of (2.4) is very perspicuous; $\Psi_\alpha^{(m)}(x,t)$ is, to within a phase factor, equal to the harmonic oscillator eigenfunction u_m displaced to the point $x_c(t)$ which follows classical motion of the harmonic oscillator with the amplitude $X \equiv 2\sigma|\alpha|$. The shape of the wavepacket remains the same all the time. The classical energy corresponding to such

oscillations is $E_c = \mu\omega^2 X^2/2 = \hbar\omega|\alpha|^2$. Since the curves $|u_m(x)|^2$ and $|\Psi_\alpha^{(m)}(x,t)|^2 = |u_m(x - x_c(t))|^2$ have the same shape, and therefore the same width, displaced number states $\Psi_\alpha^{(m)}(x,t)$ obey the familiar x–p uncertainty relation $\delta x \delta p = (m + \frac{1}{2})\hbar$ [45]. The *number phase* uncertainty product for displaced number states was examined in [49]. The probability distribution $|c_n^{(m)}|^2$ of the vibrational quanta for displaced number states exhibits characteristic oscillations which are interpreted as interference in phase space [33], similarly to the case of squeezed states [50–53]. We mention here only that the number distribution depends on the magnitude of the complex displacement parameter $|\alpha|$ (it is independent of the phase φ) and on the quantum number m. The average number of vibrational quanta, for the displaced number state $\Psi_\alpha^{(m)}(x,t)$, is $\langle n\rangle = m + |\alpha|^2$. Since m stands for the quantum number of the initial state $u_m(x)$, from which the displaced number state develops under the influence of a suitable external force (as described in the next section), we get the physical interpretation of the complex displacement parameter: its squared magnitude is equal to the average number of *excited* quanta, while its phase φ corresponds to the classical phase shift. The average energy $\langle E\rangle$ consists of two terms: the first is the energy of the initial state, while the second represents the classical oscillation energy, $\langle E\rangle = \hbar\omega(m + \frac{1}{2}) + E_c$.

Finally in this section, we mention that the displaced number states with $m \neq 0$ can, alternatively, all be generated from the $m = 0$ displaced number state (that is, from the coherent state) by repeated action of the operator $\hat{a}^\dagger e^{-i\omega t} - \alpha^*$,

$$\Psi_\alpha^{(m)}(x,t) = \frac{(\hat{a}^\dagger e^{-i\omega t} - \alpha^*)^m}{\sqrt{m!}} \Psi_\alpha^{(0)}(x,t) \tag{2.15}$$

with $m = 1, 2, 3, \ldots$. Here

$$\hat{a} \equiv \frac{\hat{x}}{2\sigma} + \frac{i\sigma}{\hbar}\hat{p} \tag{2.16}$$

denotes the familiar harmonic oscillator annihilation operator.

3. Forced harmonic oscillator

In this section we show that a single quantum oscillator driven by a suitable transient external force evolves from the initial state $u_m(x)$ to the final, displaced number state *only when the latter is modified with a suitable phase factor*. We assume a Hamiltonian of the form [48]

$$\hat{H}(t) = \frac{\hat{p}^2}{2\mu} + \frac{\mu\omega^2\hat{x}^2}{2} + \hat{H}'(t) \tag{3.1}$$

with

$$\hat{H}'(t) \equiv -\hat{x}F(t) - \hat{p}G(t) \tag{3.2}$$

representing the action of an external time-dependent force. In (3.2), $F(t)$ and $G(t)$ are given—real functions of t, different from zero only in the interval $0 < t < \tau$. Before $t = 0$ and after $t = \tau$ the Hamiltonian is that of a free oscillator. We seek a solution of the corresponding time-dependent Schrödinger equation in the form (2.1) with the expansion coefficients now evidently depending on time, $c_n = c_n(t)$. Using the matrix elements

$$\langle n|\hat{H}'(t)|m\rangle = \sqrt{m}\delta_{n\,m-1} f(t) + \sqrt{m+1}\delta_{n\,m+1} f^*(t) \tag{3.3}$$

where $f(t)$ is the complex-valued function of time,

$$f(t) \equiv -\sigma F(t) + \frac{i\hbar}{2\sigma} G(t) \tag{3.4}$$

we obtain, with the help of the usual time-dependent perturbation theory, the following system of simultaneous linear homogeneous differential equations for the functions $c_n(t)$:

$$\frac{dc_n(t)}{dt} = -\frac{i}{\hbar}\left(\sqrt{n}c_{n-1}(t)f^*(t)e^{i\omega t} + \sqrt{n+1}c_{n+1}(t)f(t)e^{-i\omega t}\right). \tag{3.5}$$

The group of equations (3.5), for all n, is *exactly* equivalent to the initial time-dependent Schrödinger equation. Appropriate inititial conditions are

$$c_m^{(0)} = 1 \qquad c_n(0) = 0 \qquad \text{for all } n \neq m. \tag{3.6}$$

Now, displaced number states have been related recently [31, 32] to the general form of the transition probability $m \to n$ for a quantum oscillator driven by an external force (this was found independently some time ago by Feynman [35, 36] and Schwinger [37]). This probability is given by $|c_n^{(m)}|^2$ with the appropriate time-dependent complex displacement parameter defined as the Fourier transform of the 'force':

$$\alpha = \alpha(t) \equiv -\frac{i}{\hbar}\int_0^t f^*(t')e^{i\omega t'}\,dt'. \tag{3.7}$$

One therefore expects that coefficients $c_n^{(m)}$ (see (2.2)), together with $\alpha(t)$ (see (3.7)), provide the required solution. This is, however, *not* the case, as one can easily see by substituting $c_n^{(m)}$ into (3.5). Only when the appropriate phase factor is incorporated (leaving the transition probability unchanged, of course),

$$c_n'^{(m)} \equiv e^{i\gamma(t)}c_n^{(m)} \tag{3.8}$$

with

$$\gamma(t) \equiv -\frac{1}{\hbar}\mathrm{Re}\left(\int_0^t \alpha(t')f(t')e^{i\omega t'}\,dt'\right) \tag{3.9}$$

do we get the required solution $c_n(t) \to c_n'^{(m)}(t)$. That the functions $c_n'^{(m)}(t)$ provide the solution of the system of differential equations (3.5), for the initial conditions (3.6), can be proved by direct substitution into (3.5) and with the help of the following two recurrence relations valid for associated Laguerre polynomials:

$$(m+k)L_m^{(k-1)}(x) - kL_m^{(k)}(x) + xL_{m-1}^{(k+1)}(x) = 0 \tag{3.10}$$

$$L_m^{(k-1)}(x) - L_m^{(k)}(x) + L_{m-1}^{(k)}(x) = 0. \tag{3.11}$$

Since the phase factor $e^{i\gamma(t)}$ is independent of m and n, equation (2.1) implies, together with $c_n(t) \to c_n'^{(m)}(t)$, that the modified wavefunction

$$\Psi_\alpha'^{(m)}(x,t) \equiv e^{i\gamma(t)}\Psi_\alpha^{(m)}(x,t) \tag{3.12}$$

and not the displaced wavefunction $\Psi_\alpha^{(m)}(x,t)$ alone, solves the given dynamic problem. The 'irrelevant' phase factor $e^{i\gamma(t)}$ is actually quite important; without it the displaced number state coefficients (2.2) do not provide the solution of the time-dependent Schrödinger equation corresponding to the Hamiltonian (3.1),(3.2). Also, this phase is not to be overlooked in the determination of the phase accumulation (considered in the next section) in the wavefunction during the time evolution.

In conclusion, if the quantum oscillator prepared initially in the eigenstate $u_m(x)$ is perturbed by a suitable transient external force, the displaced number state, modified with the phase factor $e^{i\gamma(t)}$, is obtained. In particular, by perturbing the quantum oscillator initially in the ground state one obtains the corresponding coherent state [42, 48], again modified with the phase factor $e^{i\gamma(t)}$. The considerations presented in this section show that not only the final state is a displaced number state; during the time evolution the oscillator passes continuously through a series of displaced states, corresponding to different values of the complex displacement parameter $\alpha(t)$ defined by (3.7).

4. Pancharatnam phase

In this section we first determine the geometric phase for generally *non-cyclic* evolution from an initial displaced number state at time t_1 to the final state at time t_2 ($0 \leqslant t_1 < t_2$) and, secondly, we apply the general formula to two simple examples, that of δ and harmonic forcing functions.

The first step in the determination of the Pancharatnam phase is to remove the dynamical phase (the time integral of the expectation value of the Hamiltonian) from the corresponding wavefunction [17, 19]. We define a new wavefunction:

$$\chi_\alpha^{(m)}(x,t) \equiv \exp\left(\frac{i}{\hbar}\int_0^t \langle \hat{H}(t')\rangle \, dt'\right)\Psi_\alpha'^{(m)}(x,t). \tag{4.1}$$

Replacing $\hat{H}(t)\Psi_\alpha'^{(m)}(x,t)$ with $i\hbar\partial\Psi_\alpha'^{(m)}(x,t)/\partial t$, using (3.12), (2.4), (2.14), (2.6) and

$$|\alpha(t)|^2\frac{d\varphi(t)}{dt} = \frac{d\gamma(t)}{dt} \tag{4.2}$$

we get for the expectation value of the Hamiltonian (3.1)

$$\langle \hat{H}(t)\rangle = -2\hbar\frac{d\gamma(t)}{dt} + \langle E(t)\rangle = -2\hbar\frac{d\gamma(t)}{dt} + \hbar\omega(m + |\alpha(t)|^2 + \tfrac{1}{2}) \tag{4.3}$$

so that

$$\chi_\alpha^{(m)}(x,t) = \exp\left[i\left(-\gamma(t) + (m+\tfrac{1}{2})\omega t + \omega\int_0^t |\alpha(t')|^2\,dt'\right)\right]\Psi_\alpha^{(m)}(x,t). \tag{4.4}$$

The second step is to form the inner product from two $\chi_\alpha^{(m)}$ functions corresponding to two different times t_1 and t_2. Writing this complex number in polar form,

$$\int_{-\infty}^{+\infty} \chi_\alpha^{(m)*}(x,t_1)\chi_\alpha^{(m)}(x,t_2)\,dx \equiv \rho_m \exp(i\beta_m) \qquad \rho_m > 0 \tag{4.5}$$

we get the Pancharatnam phase β_m. With the help of (2.11) we obtain

$$\rho_m = |L_m^{(0)}(|\alpha_{12}|^2)| \exp(-\tfrac{1}{2}|\alpha_{12}|^2) \tag{4.6}$$

and

$$\beta_m = \gamma(t_1) - \gamma(t_2) + \omega \int_{t_1}^{t_2} |\alpha(t)|^2 \, dt + \mathrm{Im}(\alpha(t_1)^* \alpha(t_2) \exp[i\omega(t_1 - t_2)]) + \frac{\pi}{2}[1 - \mathrm{sgn}(L_m^{(0)}(|\alpha_{12}|^2))] \tag{4.7}$$

with

$$\alpha_{12} \equiv \alpha(t_1) \exp(-i\omega t_1) - \alpha(t_2) \exp(-i\omega t_2). \tag{4.8}$$

The last term in (4.7) takes into account the possibility that $L_m^{(0)}(|\alpha_{12}|^2)$, which is an mth degree polynomial, has a negative value. Equation (4.7), together with (3.7), (3.9) and (4.8), provides the general expression for the Pancharatnam phase for displaced number states. It is seen that β_m depends on the quantum number m of the initial eigenstate.

We now focus on two simple forcing functions. Consider first the case of δ-forcing:

$$f(t) = f\delta(t - \underline{t}). \tag{4.9}$$

Here, f and $\underline{t}$ are two given constants, complex and real respectively. The corresponding displacement parameter is obtained easily from (3.7):

$$\alpha(t) = -\frac{i}{\hbar} f^* \Theta(t - \underline{t}) e^{i\omega \underline{t}} \tag{4.10}$$

with Θ denoting the Heaviside step function. The Pancharatnam phase is, in this case,

$$\beta_m = \frac{|f|^2}{\hbar^2}\{\omega[(t_2 - \underline{t})\Theta(t_2 - \underline{t}) - (t_1 - \underline{t})\Theta(t_1 - \underline{t})] - \Theta(t_1 - \underline{t})\Theta(t_2 - \underline{t}) \sin[\omega(t_2 - t_1)]\} + \frac{\pi}{2}[1 - \mathrm{sgn}(L_m^{(0)}(|\alpha_{12}|^2))]. \tag{4.11}$$

Physically more interesting is the case of the harmonic forcing function

$$f(t) = f e^{i\Omega t} = |f| e^{i(\Omega t + \phi)} \tag{4.12}$$

for $0 < t < \tau$, and zero otherwise. In (4.12), $f \equiv |f| e^{i\phi}$ is a complex constant and $\Omega (\neq \omega)$ is the forcing angular frequency. The corresponding displacement parameter is

$$\alpha(t) = \frac{|f| e^{-i\phi}}{\hbar(\omega - \Omega)} \left(1 - e^{i(\omega - \Omega)t}\right). \tag{4.13}$$

It is seen that, in the complex α-plane, $\alpha(t)$ traces a circle centred at the point $|f| e^{-i\phi}/\hbar(\omega - \Omega)$ and with the radius $|f|/\hbar|\omega - \Omega|$. It returns to the initial value $\alpha(0) = 0$ after every $2\pi/|\omega - \Omega|$ seconds. The Pancharatnam phase is, in this case,

$$\beta_m = \frac{|f|^2}{\hbar^2(\omega - \Omega)^2}\Bigg[(\omega + \Omega)(t_2 - t_1) - 2\left(\frac{\omega + \Omega}{\omega - \Omega}\right) \cos[\tfrac{1}{2}(\omega - \Omega)(t_1 + t_2)] \sin[\tfrac{1}{2}(\omega - \Omega)(t_2 - t_1)] - 4 \sin[\tfrac{1}{2}(\omega - \Omega)t_1] \sin[\tfrac{1}{2}(\omega - \Omega)t_2] \sin[\tfrac{1}{2}(\omega + \Omega)(t_2 - t_1)]\Bigg] + \frac{\pi}{2}[1 - \mathrm{sgn}(L_m^{(0)}(|\alpha_{12}|^2))]. \tag{4.14}$$

We see that, for a fixed t_1, β_m as a function of t_2 has an oscillatory contribution superimposed on a linearly increasing trend.

5. The case of cyclic evolution

In this section we consider the special case of cyclic evolution and demonstrate that the general expression for the Pancharatnam phase (4.7) reduces to the corresponding Berry phase. In the second part we examine in some detail the special case of the harmonic forcing function.

For a cyclic evolution the initial and final states of the oscillator have the same observed values. From a quantum viewpoint, the coordinate-space wavefunction $\Psi_\alpha'^{(m)}(x,t)$ is the same, except that it acquires a phase factor independent of x. Equations (2.4) and (2.5) show that this is achieved, for an evolution from t_1 to t_2, if the centres of the wavepacket in the coordinate and momentum space return to their initial values

$$x_c(t_1) = x_c(t_2) \tag{5.1}$$

$$p_c(t_1) = p_c(t_2). \tag{5.2}$$

These two conditions ensure that, for the evolution from t_1 to t_2, the momentum-space wavefunction $\Phi_\alpha'^{(m)}(p,t)$ also acquires the *same* phase factor. Since $x_c(t)$ and $p_c(t)$ follow exactly the corresponding classical motion of the representative point in phase space with, in general, varying amplitude and phase shift (see (2.6), (2.7), (2.3) and (3.7)) we see that quantal cyclic evolution for the oscillator leads, in the case of displaced number states, directly to the classical notion of cyclic evolution. Classically there is no difference between the initial and final states, and the oscillator in its final state appears as if it has not undergone any evolution. The time $t_2 - t_1$ needed to return to the initial state is the same from both quantum and classical considerations.

Equations (5.1) and (5.2) are equivalent to the following two conditions (cf (2.6) and (2.7)):

$$|\alpha(t_1)| = |\alpha(t_2)| \tag{5.3}$$

$$\omega t_1 - \varphi(t_1) = \omega t_2 - \varphi(t_2) + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{5.4}$$

Equation (5.3) requires that the magnitude of the complex displacement parameter (classically, the oscillating amplitude) returns to its initial value, while (5.4) requires that the initial and final values of the classical phase $\omega t - \varphi$ differ, at most, by an integer multiple of 2π.

Conditions (5.3) and (5.4) lead immediately to $\alpha_{12} = 0$, so that $L_m^{(0)}(|\alpha_{12}|^2) = L_m^{(0)}(0) = 1$ and therefore $\rho_m = 1$. That this must indeed be true can be seen from (4.5) since, for a cyclic evolution, the initial and final $\chi_\alpha^{(m)}$ functions are equal to within a phase factor. The Hamiltonian (3.1) is Hermitian and the evolution is unitary (norm-preserving). Additionally, (5.3) and (5.4) imply that the last two terms in (4.7) vanish and the Pancharatnam phase reduces to the familiar expression for the Berry phase [25]

$$\beta = \gamma(t_1) - \gamma(t_2) + \omega \int_{t_1}^{t_2} |\alpha(t)|^2 \, dt. \tag{5.5}$$

For a cyclic evolution the geometric phase for displaced number states becomes independent of the quantum number m and is, in fact, equal to the geometric phase for the coherent states.

Now, we focus on the case of the harmonic forcing function (4.12). Taking into account (5.5), we see that (4.14) gives

$$\beta = \frac{|f|^2(\omega+\Omega)}{\hbar^2(\omega-\Omega)^3}\{(\omega-\Omega)(t_2-t_1) - 2\cos[\tfrac{1}{2}(\omega-\Omega)(t_1+t_2)]\sin[\tfrac{1}{2}(\omega-\Omega)(t_2-t_1)]\}. \tag{5.6}$$

We also have, from (4.13), and (2.6), (2.7)

$$x_c(t) = -\frac{4\sigma|f|}{\hbar(\omega-\Omega)}\sin[\tfrac{1}{2}(\omega-\Omega)t]\sin[\tfrac{1}{2}(\omega+\Omega)t+\phi] \tag{5.7}$$

$$p_c(t) = -\frac{2|f|}{\sigma(\omega-\Omega)}\sin[\tfrac{1}{2}(\omega-\Omega)t]\cos[\tfrac{1}{2}(\omega+\Omega)t+\phi]. \tag{5.8}$$

If the angular frequencies ω and Ω are commensurable (that is, if the ratio ω/Ω is equal to a rational number n/N, with n, N positive integers), the representative point describes a closed curve in phase space (figure 1(*a*)). If the ratio ω/Ω is irrational, the curve does not close, but gradually fills (a region in) phase space (figure 1(*b*)). Since we are dealing here with a classically *non-autonomous* system, the phase trajectory can intersect with itself. We shall discuss three cases:

(i) For any t_1 and $\omega/\Omega = n/N$ (figure 1(*a*)), the time for the oscillator to return to the initial x_c and p_c values, *after a full cycle*, is

$$T \equiv t_2 - t_1 = n\frac{2\pi}{\omega} = N\frac{2\pi}{\Omega} = (n-N)\frac{2\pi}{\omega-\Omega} = (n+N)\frac{2\pi}{\omega+\Omega}. \tag{5.9}$$

Starting at the point $(x_c(t_1), p_c(t_1))$ in the phase space, the oscillator returns to the initial point after each T seconds. The last term in (5.6) vanishes and the Berry phase acquired during one cycle becomes simply

$$\beta = \frac{S}{\pi}(\omega+\Omega)T \tag{5.10}$$

with

$$S \equiv \frac{\pi|f|^2}{\hbar^2(\omega-\Omega)^2} \tag{5.11}$$

denoting the area of the circle traced by $\alpha(t)$ in the complex α-plane. In this case the coordinate-space wavefunction acquires a phase factor, $\Psi_\alpha'^{(m)}(x,t_2) = e^{i(\text{phase})}\Psi_\alpha'^{(m)}(x,t_1)$ *and* the Hamiltonian returns to its original form. Indeed, the external 'force' returns to its initial value, $f(t_2) = f(t_1)$ (see (4.12) and (5.9)), so that, from (3.1), (3.2) and (3.4), we have $\hat{H}(t_2) = \hat{H}(t_1)$.

(ii) If the starting point in the phase space happens to be one of the intersection points (A, B, ...; figure 1(*a*) or (*b*)), additional possibilities arise for the oscillator, *but not the external 'force'* $f(t)$, to return to the initial state. In the case depicted in figure 1(*a*), the routes A → O → A and A → B → O → B → A are the examples. In such cases, the oscillator returns to the original phase point $(x_c(t_1), p_c(t_1))$ after the time $t_2 - t_1$, which is

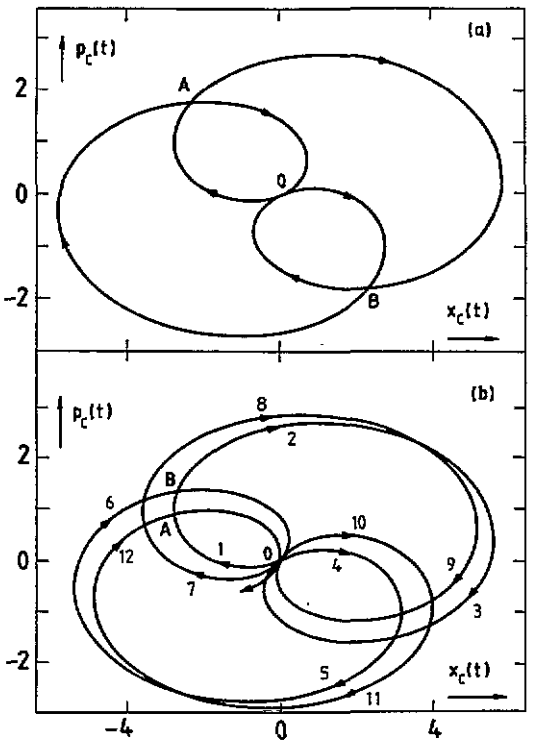

Figure 1. Phase-space curve traced by the representative point $(x_c(t), p_c(t))$ in the case of the harmonic forcing function (4.12) with $\hbar = \sigma = \omega = |f| = \phi = 1$. (*a*) Commensurable angular frequencies $\omega/\Omega = n/N = 3$. The phase-space curve is closed and cyclic evolution is possible for any starting point. (*b*) $\omega/\Omega = \pi$ = irrational. The phase-space curve $O \to 1 \to 2 \to 3 \to \ldots$ is open and gradually fills a region in phase space.

some (appropriate) integer multiple of $2\pi/(\omega + \Omega)$, and both terms in (5.6) contribute to the phase.

(iii) If $t_1 = 0$, and therefore $x_c(0) = p_c(0) = 0$, we see from (5.7) and (5.8) that the oscillator returns to the initial phase point O (figure 1(*a*) or (*b*)) after each $2\pi/(\omega - \Omega)$ seconds and the phase acquired is simply

$$\beta = 2S\left(\frac{\omega+\Omega}{\omega-\Omega}\right). \tag{5.12}$$

Note that in the (exceptional) cases (ii) and (iii) we have $\Psi_\alpha'^{(m)}(x, t_1) = e^{i(\text{phase})}\Psi_\alpha'^{(m)}(x, t_1)$ and $\hat{H}(t_2) \neq \hat{H}(t_1)$.

6. Conclusions

In this paper we have discussed the Pancharatnam phase for displaced number states of the harmonic oscillator. In particular, we reviewed the properties of these states and examined how a single quantum oscillator, driven by a suitable transient external force, evolves from the initial eigenstate $u_m(x)$ to the final, displaced number state modified with a suitable

phase factor. We emphasized the significance of this, usually neglected, phase factor for the solution of the relevant time-dependent Schrödinger equation and for the geometric phase accumulated in the wavefunction during the time evolution of the system. The general expression (4.7) for the geometric phase for a non-cyclic evolution from an initial displaced number state at time t_1 to the final state at time t_2 was derived and two examples, that of δ and harmonic forcing, were presented. The special case of cyclic evolution was subsequently discussed. In particular, the conditions (5.3) and (5.4) leading to such an evolution were derived. The relationship to the classical notion of cyclic evolution was also discussed in some detail. It was demonstrated that the general expression for the Pancharatnam phase reduces to the corresponding Berry phase. We found that, in the case of cyclic evolution, the geometric phase for displaced number states becomes independent of the quantum number m and, in fact, equal to the geometric phase for the coherent states. Finally, these general considerations were illustrated with the special case of the harmonic forcing function.

The simple harmonic oscillator occupies a privileged position in quantum physics. It is important in describing small oscillations about equilibrium positions and hence gives a description of many wave phenomena. It is in a one-to-one correspondence with a single mode of the electromagnetic field and is thus central to quantum electrodynamics and quantum optics. In recent years, it has become possible to almost perfectly isolate single quantum harmonic oscillators from their environment [54, 55], thus enabling detailed investigations of the dynamics of this simplest of all quantum systems. It is expected [33] that the displaced number states can be prepared, in the realm of quantum optics, by driving the microwave cavity field (initially prepared in a number state) of the micromaser with a classical current. This will allow, among other things, the possibility of detection of the corresponding geometric phase. Such studies will, hopefully, yield a deeper understanding of the quantum mechanics of single, isolated harmonic oscillators.

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